NEW FIXED POINT THEOREM ON COMPLEX VALUED B-METRIC SPACES

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Abstract: In this article, we extend the result recently proved by Hui-Sheng et al [5] to complete complex valued b-metric spaces.

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1 Introduction

In 1922,Banach[3] proved contraction principle which provides a technique for solving existence problem in many branches of mathematics such as mathematical analysis, computer sciences and engineering. Subsequently Banach contraction principle was generalized, extended and improved by many authors in differnt direction. In 2011, Azam et.al[1] introduced the notion of complex valued metric spaces and established some fixed point results for a pair of mapping of contraction condition satisfying a rational expression.

After the established of complex valued metric spaces, many reseachers have contributed into their work in the space. The concept of b-metric spaces is introduced by Bakhtin in [2] and used by Czerwik[4].

2 Preliminaries

In[7], Rao et. al., introduced the complex valued b-metric space and their consequences.

Definition 1 *Let X be a non-empty set and $s \geq 1$. Suppose that the mapping $d : X \times X \to \mathcal{C}$, satisfies:*
1. $0 \preceq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$
2. $d(x, y) = d(y, x)$, for all $x, y \in X$
3. $d(x, y) \preceq s\{d(x, z) + d(z, y)$, for all $x, y, z \in X\}$ Then d is called a complex valued b-metric on X and (X, d) is called a complex valued b-metric space.

Let (X, d) be a complex valued b-metric space and $S \subseteq X$. A point $x \in X$ is called interior point of a set $S \subseteq X$ whenever there exists $0 \prec r \in \mathcal{C}$ such that $B(x, r) = \{y \in X : d(x, y) \prec r\} \subseteq S$.

A point $x \in X$ is called a limit point of S whenever for every $0 \prec r \in \mathcal{C}$, $B(x, r) \cap (S - \{x\}) \neq \phi$.

A subset $S \subseteq X$ is called open whenever each element of S is an interior point of S whereas a subset $T \subseteq X$ is called closed whenever T contains all its limit points.

Definition 2 *Let (X, d) be a complex valued b-metric space and $\{x_n\}_{n\geq 1}$ be a sequence in X and $x \in X$. We say that*

1. *The sequence $\{x_n\}_{n\geq 1}$ converges to x if for every $r \in \mathcal{C}$, with $0 \prec r$ there is a positive integer n_0 such that for all $n > n_0$, $d(x_n, x) \prec r$. We write $x_n \to x$, as $n \to \infty$.*

2. *The sequence $\{x_n\}_{n\geq 1}$ is Cauchy sequence if for every $r \in \mathcal{C}$, with $0 \prec r$ there is a positive integer n_0 such that for all $n, m > n_0$, $d(x_n, x_m) \prec r$.*

3. *The metric space (X, d) is a complete complex valued b-metric space if every Cauchy sequence is convergent.*

Lemma 1 *[7] Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n, x)| \to 0$, as $n \to \infty$.*

Lemma 2 *[7] Let (X, d) be a complex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ Cauchy sequence if and only if $|d(x_n, x_{n+m})| \to 0$, as $n \to \infty$, where $m \geq 1$.*

3 Main Result

In [5], Hui-Sheng Ding et.al proved the following theorem for complete b-metric space, which states that

Theorem 1 *Let (X, d) be a complete b-metric space with $s > 1$ and B_s be the class of functions $\beta : [0, \infty) \to [0, \frac{1}{s})$ which satisfies the property that $\beta(t_n) \to \frac{1}{s}$ implies $t_n \to 0$.. If $f : X \to X$ satisfies the condition $d(f(x), f(y)) \leq \beta(d(x, y))d(x, y)$, for all $x, y \in X$ and some $\beta \in B_s$, then f has an unique fixed point $a \in X$ and for each $x \in X$, the Picard sequence $\{f^n(x)\}$ converges to a in (X, d).*

Now we extend Theorem 1 to complex valued b-metric spaces as follows,

Theorem 2 *Let (X, d) be a complete complex valued b-metric space with $s > 1$, and let $f : X \to X$ satisfies the condition $d(f(x), f(y)) \preceq \beta(|d(x, y)|)d(x, y)$, for all $x, y \in X$ and some $\beta \in B_s$ Then f has an unique fixed point $a \in X$ and for each $x \in X$, the Picard sequence $\{f^n(x)\}$ converges to a in (X, d).*
Proof: *Let $x_0 \in X$ be an arbitrary point and let $x_n = f(x_{n-1}), n = 1, 2, 3, \ldots$. If for some positive integer n such that $d(x_n, f(x_n)) = 0$, then x_n becomes a fixed point of f, which completes the proof. So we assume that $d(x_n, f(x_n)) \neq 0$ for all n. Hence we have,*

$$\begin{aligned} d(x_{n+1}, x_n) &= d(f(x_n), f(x_{n-1})) \\ &\preceq \beta(|d(x_n, x_{n-1})|)d(x_n, x_{n-1}) \\ &\preceq \frac{1}{s}d(x_n, x_{n-1}) \end{aligned}$$

Therefore

$$\begin{aligned} |d(x_{n+1}, x_n)| &\leq \frac{1}{s}|d(x_n, x_{n-1})| \\ &\leq \frac{1}{s^2}|d(x_{n-1}, x_{n-2})| \\ &\ldots, \\ &\leq \frac{1}{s^n}|d(x_1, x_0)| \end{aligned} \tag{1}$$

Since $s > 1$ and from (1), we have $|d(x_{n+1}, x_n)| \to 0$, as $n \to \infty$.

Now we have to prove that the sequence $\{x_n\}$ is a Cauchy sequence in complex valued b-metric space (X, d). Arguing by contradiction, we assume *that there is an* $\epsilon > 0$ *and the sequences* $\{p(n)\}$ *and* $\{q(n)\}$ *of positive integers such that* $p(n) > q(n) > n$, $|d(x_{p(n)}, x_{q(n)})| \geq \epsilon$ *and* $|d(x_{p(n)-1}, x_{q(n)})| < \epsilon$, *for all positive integers* n. *Now,*

$$\begin{aligned} d(x_{q(n)+1}, x_{p(n)}) &= d(f(x_{q(n)}), f(x_{p(n)-1})) \\ &\preceq \beta(|d(x_{q(n)}, x_{p(n)-1})|)d(x_{q(n)}, x_{p(n)-1}) \\ |d(x_{q(n)+1}, x_{p(n)})| &\leq \beta(|d(x_{q(n)}, x_{p(n)-1})|)|d(x_{q(n)}, x_{p(n)-1})| \\ &< \epsilon\beta(|d(x_{q(n)}, x_{p(n)-1})|) \\ Hence, \quad \frac{|d(x_{q(n)+1}, x_{p(n)})|}{\epsilon} &< \beta(|d(x_{q(n)}, x_{p(n)-1})|) \\ &< \frac{1}{s} \end{aligned}$$

whenever $p(n) > q(n) > n$, *for all positive* n. *Thus we have,*

$$\liminf_{n\to\infty} \frac{|d(x_{q(n)+1}, x_{p(n)})|}{\epsilon} \leq \liminf_{n\to\infty} \beta(|d(x_{q(n)}, x_{p(n)-1})|) \leq \frac{1}{s} \tag{2}$$

and

$$\limsup_{n\to\infty} \frac{|d(x_{q(n)+1}, x_{p(n)})|}{\epsilon} \leq \limsup_{n\to\infty} \beta(|d(x_{q(n)}, x_{p(n)-1})|) \leq \frac{1}{s} \tag{3}$$

Now,

$$\begin{aligned} \epsilon &\leq |d(x_{p(n)}, x_{q(n)})| \\ &\leq s\{|d(x_{p(n)}, x_{q(n)+1})| + |d(x_{q(n)+1}, x_{q(n)})|\} \\ from (1), we have \\ \epsilon &\leq s \liminf_{n\to\infty} |d(x_{p(n)}, x_{q(n)+1})| \\ Therefore, \quad \frac{\epsilon}{s} &\leq \liminf_{n\to\infty} |d(x_{p(n)}, x_{q(n)+1})| \end{aligned} \tag{4}$$

Combine (2), (3) and (4), we get

$$\begin{aligned} \frac{1}{s} &= \frac{1}{\epsilon}.\frac{\epsilon}{s} \\ &\leq \frac{1}{\epsilon} \liminf_{n\to\infty} |d(x_{q(n)+1}, x_{p(n)})| \\ &\leq \liminf_{n\to\infty} \beta(|d(x_{q(n)}, x_{p(n)-1})|) \\ &\leq \limsup_{n\to\infty} \beta(|d(x_{q(n)}, x_{p(n)-1})|) \\ &\leq \frac{1}{s} \end{aligned}$$

Finally, we have $\lim_{n\to\infty} \beta(|d(x_{q(n)}, x_{p(n)-1})|) = \frac{1}{s}$*, as* $n \to \infty$ *and from the definition of* β *we have,* $\lim_{n\to\infty} |d(x_{q(n)}, x_{p(n)-1})| = 0$*, as* $n \to \infty$*. Next*

$$\begin{aligned} \frac{\epsilon}{s} &\leq \frac{1}{s}|d(x_{p(n)}, x_{q(n)})| \\ &\leq \frac{1}{s}.s\{|d(x_{p(n)}, x_{p(n)-1})| + |d(x_{p(n)-1}, x_{q(n)})|\} \end{aligned}$$

allow $n \to \infty$*, we have* $\frac{\epsilon}{s} \leq 0$ *which is a contradiction, so that* $\{x_n\}$ *is a Cauchy sequence. Since* X *is complete, there is a* $x \in X$ *so that* $x_n \to x$*, as* $n \to \infty$*.*

Next we prove that this x *is a fixed point of* f*. Consider*

$$\begin{aligned} d(x, f(x)) &\preceq s\{d(x, x_{n+1}) + d(x_{n+1}, f(x))\} \\ &\preceq s\{d(x, x_{n+1}) + \beta(|d(x_n, x)|)d(x_n, x)\} \\ |d(x, f(x))| &\leq s|d(x, x_{n+1})| + s\beta(|d(x_n, x)|)|d(x_n, x)| \\ &\leq s|d(x, x_{n+1})| + |d(x_n, x)| \end{aligned}$$

which on making $n \to \infty$ *and by Lemma 1, we have* $|d(x, f(x))| \leq 0$ *and hence* $f(x) = x$*. To prove uniqueness, suppose* x *and* y *are two fixed points of* f*, then*

$$\begin{aligned} d(x, y) &= d(f(x), f(y)) \\ &\preceq \beta(|d(x, y)|)d(x, y) \\ &\preceq \frac{1}{s}d(x, y) \end{aligned}$$

as $s > 1$ *this is true only if* $x = y$*, Completes the proof.*

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