TOPOLOGIZED DOUBLE DOMINATION IN GRAPHS

S.Vimala[#1] and P.Divya [*2]

Assistant Professor, Department of Mathematics, Mother Teresa Women's University, Kodaikanal[1],

Research Scholar, Department of Mathematics, Mother Teresa Women's University, Kodaikanal[2],

email:**tvimss@gmail.com**[1],

email: **divyapandi3112@gmail.com**[2]

ABSTACT **- In this paper, the new concept, "Topologized Double Domination in graphs" is defined. Topologized Double Domination concept applied to connected graph, tree graph, path graph and complete graph. The basic properties of Topologized double dominating set, Topologized minimum double dominating set, Topologized double domination number are also analysed.**

KEYWORDS **- Topologized double dominating set, Topologized minimum double dominating set, Topologized double domination number.**

I. INTRODUCTION

The concept of Topologized graph was introduced by Antoine Vella in 2005 [1]. Antoine vella extended Topology to the Topologized graph by the S_1 space and the boundary of every vertex and edges of a graph of G. The space is called S_1 space if every singleton in the Topological space either open or closed. In 2017, Topologized graph extended to Topologized bipartite graph Topologized Hamiltonian and complete graph by Vimala.S et al [5&6]. In 1962, Ore introduced the concept of domination in "Theory of Graphs" [4]. In 1997 T. Heynes, S. Hedetniemi and P. Slater published the book, "Fundamentals of domination in graphs" [3]. After this publication there has been a rapid growth of research on this area and a wide variety of domination parameters have been introduced. In 2000 F.Harary and T.Haynes introduced double domination in double domination in graphs [7]. In this paper the concept, "Topologized domination in graphs" are introduced and also studied some of its properties.

II. PRELIMINARIES:

2.1 TOPOLOGICAL SPACE [2]:

A topological space is an ordered pair (X,τ) where X is a set, τ is a collection of subsets of X satisfying the following properties.

- Both the empty set and X are elements of τ
- The union of elements of any sub collection of τ is in τ
- The intersection of the elements of any finite sub collection of τ is in τ

If τ is a topology on X, then the pair (X,τ) is called a Topological space. The elements of τ are called open sets in X. A subset of X is said to be closed if its complement is in τ (i.e., its complement is open). A subset of X may be open, closed, both (clopen set), or neither. The empty set and X itself are always both closed and open.

2.2 TOPOLOGIZED GRAPH [1]**:**

A Topologized graph is a topological space X such that

- Every singleton is open or closed.

- $\forall x \in X$, $|\partial(x)| \leq 2$. since $\partial(x)$ is denoted by the boundary of a point x.

*2.3 DOMINATING SET [4]***:**

A subset S of V(G) is said to be dominating set if for every vertex v in $V(G)-S$, there is a vertex u in S such that u is adjacent to v. That is a vertex v of G is in S or is adjacent to some vertex of S

2.4 MINIMUM DOMINATING SET [4]:

A dominating set with least number of vertices is called minimum dominating set. It is denoted as γ set of the graph G.

2.5 DOMINATION NUMBER [4]:

The number of vertices in a minimum dominating set is called domination number of the graph G. It is denoted by γ(G).

2.6 DOUBLE DOMINATING SET [7&8]:

A subset S of V is a Double Dominating set of G, or DDS, if for every vertex v ∈ V, |N[v]∩S| ≥ 2, that is, v is in S and has at least one neighbor in S or v is in V−S and has at least two neighbors in S. So a Double Dominating set dominates every vertex in G at least twice

2.7 MINIMUM DOUBLE DOMINATING SET [7&8]:

A Double Dominating set with least number of vertices is called Minimum Double Dominating set. It is denoted by γ_{dds} .

2.8 DOUBLE DOMINATING NUMBER [7&8]:

A Double Dominating Number of $\gamma_{dds}(G)$ of G is the minimum cardinality of a double dominating set of G.

Result [6]:

If G is a path Pn , then it is a Topologized graph, for n ≥ 1.

Result [6]:

If G is circuit Cn, then it is a Topologized graph, for n≥1.

Result [6]:

If G is a Complete graph Kn, then it is not a Topologized graph, for n≥4.

Result [5]:

$K_{1,n}$ is a Topologized graph (for 0<n≤2).

III. MAIN RESULT:

3.1 TOPOLOGIZED DOUBLE DOMINATING SET:

A subset S of V is a Topologized Double Dominating set of G, or DDS, if for every vertex v ∈ V, |N[v]∩S| ≥ 2, that is, v is in S and has at least one neighbor in S or v is in V−S and has at least two neighbors in S. So a Topologized Double Dominating set dominates every vertex in G at least twice.

EXAMPLE 3.2:

Let G be a complete graph K_3 with 3 vertices and 3 edges.

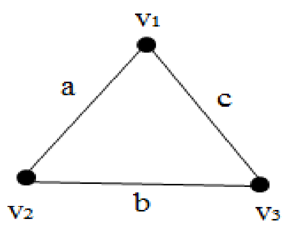

Fig. 1

G is also a Topologized graph. Topologized minimum double dominating sets are D = {v_1, v_2}such that $\tau\gamma_{dds}(G) = 2$.

3.3 TOPOLOGIZED MINIMUM DOUBLE DOMINATING SET:

A Topologized Double Dominating set with least number of vertices is called Topologized Minimum Double Dominating set. It is denoted by $\tau\gamma_{dds}$.

EXAMPLE 3.4:

From figure 1, Topologized minimum dominating set is D = {v_1}such that $\tau\gamma(G) = 1$

3.5 TOPOLOGIZED DOUBLE DOMINATING NUMBER:

A Topologized Double Dominating Number of $\tau\gamma_{dds}(G)$ of G is the minimum cardinality of a topologized double dominating set of G.

EXAMPLE 3.6:

From figure 1, the Topologized double dominating sets are Topologized minimum double dominating sets. And the Topologized double dominating number $\tau\gamma_{dds}(G) = 2$

PREPOSITION 3.7:

(1) If D is a topologized minimum double dominating set of a tree T, then every leaf of T belongs to D
Note: a vertex of degree 1 in T as a leaf of T

(2) If a topologized graph G has no isolated vertices, then $\tau\gamma(G) \le \tau\gamma_{rs}(G)$

(3) For any complete graph *Kp* , with $p \geq 2$ vertices, $\gamma_{dds} = \tau\gamma_{dds}(K_p)$=2.

(4) If G is a graph with at most degree two, then $\tau\gamma_{rs}(G) \ge 2$

THEOREM 3.8:

Let T be a tree with n vertices and atmost degree two. Then $\tau\gamma_{dds}(G) \ge \frac{n+1}{3}$.

PROOF:

Let X be a topological space with topology τ defined by $V \cup E$. Since every singleton set is open or closed and G is a tree graph with atmost degree two, the boundary of each vertices is less than or equal to two which implies that G is a Topologized graph. Since T has n-1 edges, $m = n - 1$. From the above Theorem, $\tau\gamma_{dds}(G) \ge \frac{n+1}{3}$.

THEOREM 3.9:

Let G be a graph without isolated vertices. Then $\tau\gamma_{dds}(G) \le \tau\gamma_{dds}(G-e) \le \tau\gamma_{dds}(G)+2$ for every non-pendent edge $e \in E(G)$.

PROOF:

Let e = xy be a non-pendent edge. Clearly, every $\tau\gamma_{dds}(G-e)$ set is a DDS of G and so $\tau\gamma_{dds}(G) \le \tau\gamma_{dds}(G-e)$. Now let S be a $\tau\gamma_{dds}(G)$ set. If $S \cap \{x, y\} = \phi$, then S is a DDS of G-e and hence $\tau\gamma_{dds}(G-e) \le \tau\gamma_{dds}(G)$. Assume now, without loss of generality, that $S \cap \{x, y\} =$ {y}. Then since x has two neighbors in S, $S \cup \{x\}$ is a DDS of G-e implying that $\tau\gamma_{dds}(G-e) \le \tau\gamma_{dds}(G)+1$. Finally assme that {x,y} $\subseteq S$.

We examine three cases.

If each x and y has degree at least two in G[s], then since e is a non pendent edge, S remains a DDS of G-e and so $\tau\gamma_{dds}(G-e) \le \tau\gamma_{dds}(G)$. Assume that both x and y are pendent vertices in G[s]. since e = xy is non-pendent edge each of x and y has a neighbor in v-S. let x^1,$y^1 \in$ v- S be a neighbors of x and y respectively. Then $S \cup \{x^1, y^1\}$is a DDS of G-e and so $\tau\gamma_{dds}(G-e) \le \tau\gamma_{dds}(G)+2$ finally assume without loss of generality, that x is a vertex of degree one in G[s] and y has degree at least two in G[s].Since xy is a non-pendent edge, let x$\in$ v-S be any neighbor of x. Then S $\cup$ {x^1}is a DDS of G-e and so $\tau\gamma_{dds}(G-e) \le \tau\gamma_{dds}(G)+1$

THEOREM 3.10:

For any Topologized graph G, $\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \le \tau\gamma_{dds}(G)$.

PROOF:

By the Theorem, $\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \le \tau\gamma(G) \le n-\Delta(G)$, then $\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \le \tau\gamma(G) \le \tau\gamma_{dds}(G)$.Thus, $\left\lceil \frac{n}{1+\Delta(G)} \right\rceil \le \tau\gamma_{dds}(G)$.

THEOREM 3.11:

Let G be a connected graph with n vertices and m edges and atmost degree two and if $\tau\gamma(G) = \tau\gamma_{dds}(G)$. Then $\tau\gamma_{dds}(G) \ge \frac{1}{3}(2n-m)$.

PROOF:

Let D_1 and D_2 be Topologized minimum dominating and Topologized double dominating sets of G respectively. Then $m \ge (n-2\tau\gamma(G)) + \tau\gamma(G) = 2n-3\tau\gamma(G)$ implies that $\gamma(G) \ge \frac{1}{3}(2n-m)$. Since $\tau\gamma(G) = \tau\gamma_{dds}(G)$ implies that $\tau\gamma_{dds}(G) \ge \frac{1}{3}(2n-m)$.

THEOREM 3.12:

If T is a tree of order n ≥ 1, then $\tau\gamma_{dds}(T) \geq \lceil (n+2)/3 \rceil$

PROOF:

We use induction over n. It is easy to check that the result is true for all trees of order n ≤ 5. Suppose, therefore that the result is true for all trees of order less than n, where n ≥ 6. Let $\tau\gamma_{dds} = \min\{\tau\gamma_{dds}(T)/T$ is a tree of order n }. We will show that $\tau\gamma_{dds} \geq \lceil (n+2)/3 \rceil$.

Let ₣ = {T/T is a tree of order n such that $\tau\gamma_{dds}(T) = \tau\gamma_{dds}$ } among all trees in ₣, Let T be chosen so that the sum s(T) of the degrees of its vertices of degree at least 3 is a minimum. If s(T) = 0 then $T \cong P_n$, and so $\tau\gamma_{dds} = \tau\gamma_{dds}(P_n) \geq \lceil (n+2)/3 \rceil$. Suppose therefore that s(T) ≥ 1. Let S be topologized minimum dominating of T.

THEOREM 3.13:

For any connected (p,q) graph G

$$2\tau\gamma_{dds}(G) \leq \tau\gamma_{dds}(G) + 2$$

PROOF:

Suppose S = {$v_1, v_2, v_3, \ldots v_n$} $\subseteq$ V(G) be the minimal set of vertices which cover all the vertices, such that dis(u,v) ≥ 3 for all{u,v} $\in$ S. Then S forms a minimal set of G. Further if for every v $\in$ V(G), there exists at least two vertices {u,w} $\in$ S such that $\forall$ u,v,N(v) and N(u) belongs to V(G) – S. Then S itself is a topologized double dominating set of G otherwise, there exists at least one vertex x $\in$ N(S) such that S $\cup$ {x} = D forms topologized double dominating set of G. Since for any graph with n ≥ 2, $\tau\gamma_{dds}(G) \geq 2$. Therefore it follows that $|S| \leq (|D| + 2)/2$.

Clearly, $2\tau\gamma_{dds}(G) \leq \tau\gamma_{dds}(G) + 2$.

THEOREM 3.14:

For any connected graph G with n ≥ 3 vertices, $\tau\gamma_{dds}(G) \leq \lceil n/2 \rceil + 1$

PROOF:

For n ≥ 2 topologized double dominating set is not less than or equal $\lceil n/2 \rceil$. For n ≥ 3, we prove the result by induction process. Suppose n = $|V| \leq 3$ in G, then $\tau\gamma_{dds}(G) = \lceil n/2 \rceil$. Assume that the result is true for any graph with n vertices. Let G be a graph with n+1 vertices. Then by induction hypothesis, it follows that $\tau\gamma_{dds}(G) = \lceil n+1/2 \rceil$.

Hence the result is true for all graphs with n ≥ 3 vertices by induction process.

IV. CONCLUSION:

In this paper we obtain the result of the topologized double domination number for some standard graphs and also establish some general results. The authors can also extend to other domination parameters.

REFERENCES:

[1]. Antoine Vella., “A Fundamendally Topological perspective on graph theory”. Ph.D., Thesis Waterloo, Ontario,Canada; 2005.

[2]. Brown, Renold, “Topology and Grouoids”, booksurge (2006) ISBN 1 – 4196-2722-8 (3rd edition of differently titled books).

[3]. Haynes T.W., Hedetniemi S.T., and Slaster P.J., “ Fundamentals of Domination in graphs”, Marcel Dekker Newyorks (1998).

[4]. Ore O., “ Theory of Graphs”, Amer. Math. Soc. Colloq. Publ.,38, Providence, (1962).

[5]. Vimala .S and Kalpana S., "Topologized Bipartite Graph", Asian Research Jornal Of Mathematics 4(1):1-10, 2017; Article no.ARJOM.33107.

[6]. Vimala S and Priyanka K., "Topologized Hamiltonian and Complete graph", Asian Research Journal of Mathematics 4(2): 1-10,2017; Article no. ARJOM.33109.

[7]. Harary F, Haynes T, "Double Domination in graphs", Ars combinatorial 55 (2000), 201 – 213.

[8]. Derrick Wayney Thacker, "Double Domination Edge Critical graphs", May(2006)